

The growth constants of lattice trees and lattice animals in high dimensions

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Abstract

We prove that the growth constants for nearest-neighbour lattice trees and lattice (bond) animals on the integer lattice \mathbb{Z}^d are asymptotic to $2de$ as the dimension goes to infinity, and that their critical one-point functions converge to e . Similar results are obtained in dimensions $d > 8$ in the limit of increasingly spread-out models; in this case the result for the growth constant is a special case of previous results of M. Penrose. The proof is elementary, once we apply previous results of T. Hara and G. Slade obtained using the lace expansion.

1 The main result

We define two different regular graphs with vertex set \mathbb{Z}^d , as follows. The *nearest-neighbour* graph has edge set consisting of pairs $\{x, y\}$ with $\|x - y\|_1 = 1$. The *spread-out* graph has edge set consisting of pairs $\{x, y\}$ with $0 < \|x - y\|_\infty \leq L$, with $L \geq 1$ fixed. These graphs have degrees $2d$ and $(2L + 1)^d - 1$, respectively. Often we discuss both graphs simultaneously, and use K to denote the degree in either case. Also, we will write $\lim_{K \rightarrow \infty}$ to simultaneously denote the limit as $d \rightarrow \infty$ for the nearest-neighbour case, and the limit as $L \rightarrow \infty$ for the spread-out case.

On either graph, a *lattice animal* is a finite connected subgraph, and a *lattice tree* is a finite connected subgraph without cycles. These very natural combinatorial objects are also fundamental in polymer science [13]. We denote the number of lattice animals containing n bonds and containing the origin of \mathbb{Z}^d by a_n , and the number of lattice trees containing n bonds and containing the origin of \mathbb{Z}^d by t_n . Standard subadditivity arguments [14, 15] provide the existence of the *growth constants*

$$\tau = \lim_{n \rightarrow \infty} t_n^{1/n}, \quad \alpha = \lim_{n \rightarrow \infty} a_n^{1/n}. \quad (1.1)$$

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The growth constants of course depend on d , and for the spread-out model, also on L . The *one-point functions*

$$g(z) = \sum_{n=0}^{\infty} t_n z^n \quad \text{and} \quad g^{(a)}(z) = \sum_{n=0}^{\infty} a_n z^n \quad (1.2)$$

have radii of convergence $z_c = \tau^{-1}$ and $z_c^{(a)} = \alpha^{-1}$, respectively.

We will rely on a result obtained by Hara and Slade [7] using the lace expansion, but we will not need any details about the lace expansion in this paper. It is shown in [7] that $g(z_c)$ and $g^{(a)}(z_c^{(a)})$ are finite (in fact, at most 4) for the nearest-neighbour model in sufficiently high dimensions, and for the spread-out model in dimensions $d > 8$ if L is sufficiently large, and that, in these two limits, z_c and $z_c^{(a)}$ obey the equations

$$\lim_{K \rightarrow \infty} K z_c g(z_c) = \lim_{K \rightarrow \infty} K z_c^{(a)} g^{(a)}(z_c^{(a)}) = 1. \quad (1.3)$$

This is discussed for the nearest-neighbour model in [6] (see, in particular, [6, (1.31)]), and the same considerations apply for the spread-out model. In fact, much more is known [19].

Our main result is the following theorem. The asymptotic relation in its statement means that the limit of the ratio of left- and right-hand sides is equal to 1.

Theorem 1. *For the nearest neighbour model as $d \rightarrow \infty$, and for the spread-out model in dimensions $d > 8$ as $L \rightarrow \infty$,*

$$\tau \sim K e \quad \text{and} \quad \alpha \sim K e, \quad (1.4)$$

and, in these same limits,

$$\lim_{K \rightarrow \infty} g(z_c) = \lim_{K \rightarrow \infty} g^{(a)}(z_c^{(a)}) = e. \quad (1.5)$$

To our knowledge, Theorem 1 is new for the nearest-neighbour model. The proof of Theorem 1 is the same for both the nearest-neighbour and spread-out models. No bound on the rate of convergence is obtained here for either (1.4) or (1.5). Given (1.3), the statements $\tau \sim K e$ and $g(z_c) \rightarrow e$ are equivalent, as are the statements $\alpha \sim K e$ and $g^{(a)}(z_c^{(a)}) \rightarrow e$.

Stronger results than (1.4) have been obtained by Penrose [18] for the spread-out model using a completely different method of proof, without restriction to $d > 8$ and with the error estimate

$$\frac{K^K}{(K-1)^{K-1}} - O(K^{5/7} \log K) \leq \tau \leq \alpha \leq \frac{K^K}{(K-1)^{K-1}} \quad (1.6)$$

in all dimensions $d \geq 1$. Both the right- and left-hand sides of (1.6) are of course asymptotic to $K e$ as $K \rightarrow \infty$. When combined with (1.3), (1.5) then follows from (1.6) for the spread-out model in dimensions $d > 8$.

Much stronger results than (1.4) have been obtained for the closely related models of self-avoiding walks and percolation. Let c_n denote the number of n -step self-avoiding walks starting at the origin. For nearest-neighbour self-avoiding walks, it was proved in [8] that the connective constant $\mu = \lim_{n \rightarrow \infty} c_n^{1/n}$ has an asymptotic expansion $\mu \sim \sum_{i=-1}^{\infty} m_i (2d)^{-i}$ (as $d \rightarrow \infty$), with $m_i \in \mathbb{Z}$ for all i . The first thirteen coefficients in this expansion are now known [2], and it appears likely that the series $\sum_i m_i x^i$ has radius of convergence equal to zero. It may however be Borel

summable, and a partial result in this direction is given in [5]. Some related results for nearest-neighbour bond percolation are obtained in [8, 11], and for spread-out models of percolation and self-avoiding walks in [10, 17, 18].

The behaviour of τ and α for the nearest-neighbour model, as $d \rightarrow \infty$, has been extensively studied in the physics literature. For τ , the expansion

$$\tau = \sigma e \exp \left(-\frac{1}{2} \frac{1}{\sigma} - \frac{8}{3} \frac{1}{\sigma^2} - \frac{85}{12} \frac{1}{\sigma^3} - \frac{931}{20} \frac{1}{\sigma^4} - \frac{2777}{10} \frac{1}{\sigma^5} + \dots \right) \quad \text{where } \sigma = 2d - 1 \quad (1.7)$$

is reported in [4], but without a rigorous estimate for the error term. This raises the question of whether there exists an asymptotic expansion for τ of the form $e \sum_{i=-1}^{\infty} r_i (2d)^{-i}$, with $r_i \in \mathbb{Q}$. For α , the series

$$\begin{aligned} \alpha = \sigma e \exp \left(-\frac{1}{2} \frac{1}{\sigma} - \left(\frac{8}{3} - \frac{1}{2e} \right) \frac{1}{\sigma^2} - \left(\frac{85}{12} - \frac{1}{4e} \right) \frac{1}{\sigma^3} - \left(\frac{931}{20} - \frac{139}{48e} - \frac{1}{8e^2} \right) \frac{1}{\sigma^4} \right. \\ \left. - \left(\frac{2777}{10} + \frac{177}{32e} - \frac{29}{12e^2} \right) \frac{1}{\sigma^5} + \dots \right) \end{aligned} \quad (1.8)$$

was derived in [9, 16], again without a rigorous error estimate; here the role of the transcendental number e is more delicate. Theorem 1 provides a rigorous confirmation of the leading terms in (1.7)–(1.8).

2 Main steps in the proof

We define

$$z_0 = \frac{1}{Ke}. \quad (2.1)$$

Since $\tau \leq \alpha$ by definition, the critical points obey

$$z_c \geq z_c^{(a)}. \quad (2.2)$$

In fact, the strict inequality $\tau < \alpha$ is known [13]. The proof of Theorem 1 uses the following two ingredients. The content of the first is that $z_c^{(a)} \geq z_0$, or, equivalently, that $\alpha \leq Ke$. This fact is presumably well-known, though we did not find an explicit proof in the literature. Klarner [14] proves that for 2-dimensional nearest-neighbour site animals the growth constant is at most $27/4 = 3^3/2^2$ and Penrose [18] states that this can be generalised to the upper bound $\alpha \leq K^K/(K-1)^{K-1} \sim Ke$ for bond animals on an arbitrary regular graph. We will provide an elementary proof that $\alpha \leq Ke$ in Lemma 2 below, both to keep self-contained and because elements of the proof are also useful elsewhere in our approach.

Lemma 2. *In all dimensions $d \geq 1$, and for the nearest-neighbour or spread-out models,*

$$z_c \geq z_c^{(a)} \geq z_0 = \frac{1}{Ke}. \quad (2.3)$$

Proposition 3. *For the nearest-neighbour model, or for the spread-out model in dimensions $d \geq 1$,*

$$\lim_{K \rightarrow \infty} g(z_0) = e. \quad (2.4)$$

Proof of Theorem 1. We will prove that, under the hypotheses of Theorem 1,

$$\lim_{K \rightarrow \infty} g(z_c) = e. \quad (2.5)$$

It then follows from (1.3) that $z_c \sim z_0$. Lemma 2 then implies that $z_c^{(a)} \sim z_0$, and finally (1.3) implies that $\lim_{K \rightarrow \infty} g(z_c^{(a)}) = e$. Thus Theorem 1 will follow, once we prove (2.5). By Proposition 3 and (1.3),

$$\lim_{K \rightarrow \infty} (K z_c g(z_c) - e^{-1} g(z_0)) = 0. \quad (2.6)$$

This can be rewritten as

$$\lim_{K \rightarrow \infty} [K(z_c - z_0)g(z_c) + e^{-1}(g(z_c) - g(z_0))] = 0. \quad (2.7)$$

By Lemma 2 and the monotonicity of g , both terms in the limit are non-negative, and therefore

$$\lim_{K \rightarrow \infty} (g(z_c) - g(z_0)) = 0. \quad (2.8)$$

With Proposition 3, this gives (2.5) and completes the proof. \square

It remains to prove Lemma 2 and Proposition 3.

3 The proof completed

We will make use of the following mean-field model (see [1, 19]), which is related to the Galton–Watson branching process with critical Poisson offspring distribution. In other developments, the connection with the mean-field model is reflected by the super-Brownian scaling limits proved for lattice trees in high dimensions [3, 12].

Let \mathcal{T}_n denote the set of n -edge rooted plane trees [20], and let $\mathcal{T} = \cup_{n=0}^{\infty} \mathcal{T}_n$. Given $T \in \mathcal{T}$, we consider mappings $\varphi : T \rightarrow \mathbb{Z}^d$ with the property that φ maps the root to the origin, and maps each other vertex of T to a neighbour of its parent (nearest-neighbour or spread-out, depending on the setting); the set of such mappings is denoted $\Phi(T)$. There is no self-avoidance constraint. By definition, for $T \in \mathcal{T}_n$, the cardinality of $\Phi(T)$ is K^n . We may interpret the image of T under φ as a multigraph without self-lines, and we refer to the pair (T, φ) as a *mean-field* configuration. The set of all mean-field configurations (T, φ) with $T \in \mathcal{T}_n$ is denoted \mathcal{M}_n .

Let ξ_i denote the forward degree of a vertex $i \in T$; this is the degree of the root when i is the root of T and otherwise it is the degree of i minus 1. For $n \geq 0$, let

$$f_n = \sum_{(T, \varphi) \in \mathcal{M}_n} \prod_{i \in T} \frac{1}{\xi_i!} = K^n \sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!}, \quad (3.1)$$

where the second equality follows from the fact that $\Phi(T)$ has K^n elements. Let $|T|$ denote the number of edges in T . Then

$$\sum_{n=0}^{\infty} f_n z^n = (Kz)^{-1} \sum_{T \in \mathcal{T}} (Kz)^{|T|+1} \prod_{i \in T} \frac{1}{e \xi_i!}. \quad (3.2)$$

Moreover, since $\mathbb{P}(T) = \prod_{i \in T} (e \xi_i!)^{-1}$ is the probability that T arises as the family tree of a Galton–Watson branching process with critical Poisson offspring distribution, it follows from (3.2) that

$$\sum_{n=0}^{\infty} f_n z_0^n = e \sum_{T \in \mathcal{T}} \mathbb{P}(T) = e. \quad (3.3)$$

The relation with the critical Poisson branching process can easily be exploited further (see, e.g., [1, Theorem 2.1]) to obtain

$$\sum_{n=0}^{\infty} f_n z^n = (Kz)^{-1} \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} (Kz)^n. \quad (3.4)$$

The series on the right-hand side converges if and only if $|Kz| \leq 1$, by Stirling's formula, and hence

$$\lim_{n \rightarrow \infty} f_n^{1/n} = Ke = \frac{1}{z_0}. \quad (3.5)$$

Let \mathcal{L}_n denote the set of n -bond lattice trees containing the origin; its cardinality is t_n . We will use the fact, proved in [1, (5.5)], that for every $L \in \mathcal{L}_n$,

$$\sum_{(T, \varphi) \in \mathcal{M}_n : \varphi(T) = L} \prod_{i \in T} \frac{1}{\xi_i!} = 1. \quad (3.6)$$

The proof of (3.6) in [1] is given for the nearest-neighbour model, but it applies without change also to the spread-out model. By summing (3.6) over $L \in \mathcal{L}_n$, we obtain

$$t_n \leq f_n, \quad (3.7)$$

and hence $\tau \leq \lim_{n \rightarrow \infty} f_n^{1/n} = Ke$. This gives the inequality $z_c \geq z_0$, which is weaker than the inequality $z_c^{(a)} \geq z_0$ that we seek in Lemma 2.

Proof of Lemma 2. The inequality $z_c \geq z_c^{(a)}$ follows from $t_n \leq a_n$, and the equality $z_0 = (Ke)^{-1}$ holds by definition, so it suffices to prove that $z_c^{(a)} \geq z_0$. By (3.5), for this it suffices to prove that

$$a_n \leq f_n. \quad (3.8)$$

To prove this, we adapt the proof of (3.6) from [1].

The first step involves a unique determination of a tree structure within a lattice animal. For this, we order all bonds in the infinite lattice lexicographically. Also, we regard a bond as an arc joining the vertices of its endpoints, and we order the two halves of this arc as minimal and

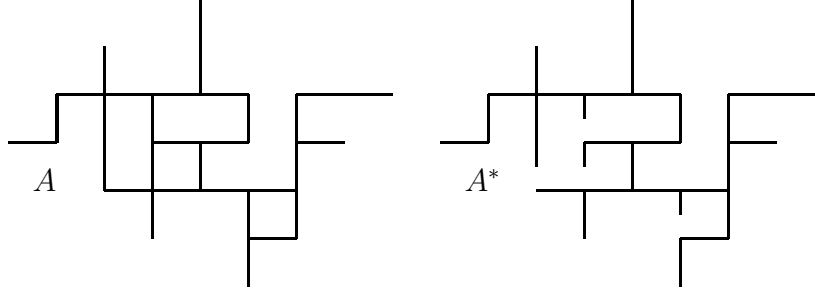


Figure 1: A lattice animal A and its associated cut-tree A^* .

maximal. These orderings are fixed once and for all. Given a lattice animal A , suppose that it contains c cycles. Choose the minimal bond whose removal would break a cycle, and remove its minimal half from the animal. Repeat this until no cycles remain. The result is a kind of lattice tree, which we will call the *cut-tree* A^* , in which c leaves are endpoints of half edges. See Figure 1. Let \mathcal{A}_n denote the set of n -bond lattice animals that contain the origin. Let \mathcal{A}_n^* denote the set of n -bond cut-trees that can be produced from a lattice animal in \mathcal{A}_n by this procedure. By construction, lattice animals and cut-trees are in one-to-one correspondence, so \mathcal{A}_n^* has cardinality a_n .

We may regard the edges of $T \in \mathcal{T}$ as directed away from the root, and we write a directed edge as (i, i') . Given $A^* \in \mathcal{A}_n^*$ and $(T, \varphi) \in \mathcal{M}_n$, we say that $\varphi(T) = A^*$ if (i) each bond in A is the image of a unique edge in T under φ , and if, in addition, (ii) if (b^+, b^-) is a directed bond in A from which the half-bond containing b^- is removed in A^* , and if the edge of T that is mapped by φ to (b^+, b^-) is (i, i') , then i' is a leaf of T . Roughly speaking, the condition $\varphi(T) = A^*$ means that the mapping φ “folds” T over A^* in such a way that the tree structure of T is preserved in A^* . We claim that for every $A^* \in \mathcal{A}_n^*$,

$$\sum_{(T, \varphi) \in \mathcal{M}_n : \varphi(T) = A^*} \prod_{i \in T} \frac{1}{\xi_i!} = 1. \quad (3.9)$$

This implies that

$$a_n = \sum_{(T, \varphi) \in \mathcal{M}_n : \varphi(T) \in \mathcal{A}_n^*} \prod_{i \in T} \frac{1}{\xi_i!} \leq \sum_{(T, \varphi) \in \mathcal{M}_n} \prod_{i \in T} \frac{1}{\xi_i!} = f_n, \quad (3.10)$$

which is the required inequality (3.8). Thus it suffices to prove (3.9).

To prove (3.9), we adapt the proof of (3.6) from [1], as follows. Let b_0 be the degree of 0 in A^* , and given a nonzero vertex $x \in A^*$, let b_x be the degree of x in A^* minus 1 (the forward degree of x). Then the set $\{b_x : x \in A^*\}$ (with repetitions) must be equal to the set $\{\xi_i : i \in T\}$ (with repetitions) for any T such that $\varphi(T) = A^*$. Defining $\nu(A^*)$ to be the cardinality of $\{(T, \varphi) : \varphi(T) = A^*\}$, (3.9) is therefore equivalent to

$$\nu(A^*) = \prod_{x \in A^*} b_x!. \quad (3.11)$$

We prove (3.11) by induction on the number N of generations of A^* , i.e., the number of bonds or half-bonds in the longest self-avoiding path in A^* starting from the origin. The identity (3.11) clearly holds if $N = 0$. Our induction hypothesis is that (3.11) holds if there are $N - 1$ or fewer generations. Suppose A^* has N generations, and let $A_1^*, \dots, A_{b_0}^*$ denote the cut-trees resulting by deleting from A^* the origin and all bonds and half-bonds incident on the origin. We regard each A_a^* as rooted at the non-zero vertex in the corresponding deleted bond. It suffices to show that $\nu(A^*) = b_0! \prod_{a=1}^{b_0} \nu(A_a^*)$, since each A_a^* has fewer than N generations.

To prove this, we note that each pair (T, φ) with $\varphi(T) = A^*$ induces a set of (T_a, φ_a) such that $\varphi_a(T_a) = A_a^*$. This correspondence is $b_0!$ to 1, since (T, φ) is determined by the set of (T_a, φ_a) , up to permutation of the branches of T at its root. This proves $\nu(A^*) = b_0! \prod_{a=1}^{b_0} \nu(A_a^*)$, and completes the proof of the lemma. \square

Lemma 4. *For the nearest-neighbour or spread-out models (the latter in all dimensions $d \geq 1$), for each fixed $n \geq 0$,*

$$\lim_{K \rightarrow \infty} \frac{t_n}{K^n} = \sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!}. \quad (3.12)$$

Proof. By (3.6),

$$t_n = \sum_{(T, \varphi) \in \mathcal{M}_n: \varphi(T) \in \mathcal{L}_n} \prod_{i \in T} \frac{1}{\xi_i!} = \sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!} \sum_{\varphi \in \Phi(T): \varphi(T) \in \mathcal{L}_n} 1. \quad (3.13)$$

Given $T \in \mathcal{T}_n$, the cardinality of $\Phi(T)$ is K^n , so there are at most K^n nonzero terms in the above sum over φ . On the other hand, there are at least $K(K-1) \cdots (K-n+1)$ nonzero terms. To see this, consider the mapping φ of T to proceed in a connected fashion to map the edges of T one by one to bonds in \mathbb{Z}^d , starting from the root. The first edge of T can be mapped to any one of K possible bonds. The second edge of T includes one of the vertices of the first edge, and to avoid the image of the other vertex of the first edge, it can be mapped to any one of $K-1$ possible edges. In this way, as φ proceeds from the root to map vertices of T into \mathbb{Z}^d , the restriction that the image contain $n+1$ distinct vertices allows K choices for the first bond, $K-1$ choices for the second bond, at least $K-2$ for the third, at least $K-3$ for the fourth, and so on. This implies that

$$K(K-1) \cdots (K-n+1) \sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!} \leq t_n \leq K^n \sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!}, \quad (3.14)$$

and the desired conclusion follows. \square

Proof of Proposition 3. By (3.7), (3.1) and (2.1),

$$t_n z_0^n \leq f_n z_0^n = e^{-n} \sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!}, \quad (3.15)$$

which is independent of K . Also, by (3.3), $\sum_{n=0}^{\infty} f_n z_0^n = e$. Hence, by Lemma 4 and the dominated convergence theorem, we have

$$\lim_{K \rightarrow \infty} g(z_0) = \sum_{n=0}^{\infty} \lim_{K \rightarrow \infty} t_n z_0^n = \sum_{n=0}^{\infty} \left(\sum_{T \in \mathcal{T}_n} \prod_{i \in T} \frac{1}{\xi_i!} \right) e^{-n} = \sum_{n=0}^{\infty} f_n z_0^n = e. \quad (3.16)$$

\square

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References

- [1] C. Borgs, J.T. Chayes, R. van der Hofstad, and G. Slade. Mean-field lattice trees. *Ann. Combinatorics*, **3**:205–221, (1999). MR1772346
- [2] N. Clisby, R. Liang, and G. Slade. Self-avoiding walk enumeration via the lace expansion. *J. Phys. A: Math. Theor.*, **40**:10973–11017, (2007). MR2396212
- [3] E. Derbez and G. Slade. The scaling limit of lattice trees in high dimensions. *Commun. Math. Phys.*, **193**:69–104, (1998). MR1620301
- [4] D.S. Gaunt and P.J. Peard. $1/d$ -expansions for the free energy of weakly embedded site animal models of branched polymers. *J. Phys. A: Math. Gen.*, **33**:7515–7539, (2000). MR1802107
- [5] B.T. Graham. Borel-type bounds for the self-avoiding walk connective constant. *J. Phys. A: Math. Theor.*, **43**:235001, (2010). MR2646672
- [6] T. Hara. Decay of correlations in nearest-neighbor self-avoiding walk, percolation, lattice trees and animals. *Ann. Probab.*, **36**:530–593, (2008). MR2393990
- [7] T. Hara and G. Slade. On the upper critical dimension of lattice trees and lattice animals. *J. Stat. Phys.*, **59**:1469–1510, (1990). MR1063208
- [8] T. Hara and G. Slade. The self-avoiding-walk and percolation critical points in high dimensions. *Combin. Probab. Comput.*, **4**:197–215, (1995). MR1356575
- [9] A.B. Harris. Renormalized $(1/\sigma)$ expansion for lattice trees and localization. *Phys. Rev. B*, **26**:337–366, (1982). MR0668821
- [10] R. van der Hofstad and A. Sakai. Critical points for spread-out self-avoiding walk, percolation and the contact process. *Probab. Theory Related Fields*, **132**:438–470, (2005). MR2197108
- [11] R. van der Hofstad and G. Slade. Expansion in n^{-1} for percolation critical values on the n -cube and Z^n : the first three terms. *Combin. Probab. Comput.*, **15**:695–713, (2006). MR2248322
- [12] M. Holmes. Convergence of lattice trees to super-Brownian motion above the critical dimension. *Electr. J. Probab.*, **13**:671–755, (2008). MR2399294
- [13] E. J. Janse van Rensburg. *The Statistical Mechanics of Interacting Walks, Polygons, Animals and Vesicles*. Oxford University Press, Oxford, (2000). MR1858028
- [14] D.A. Klarner. Cell growth problems. *Canad. J. Math.*, **19**:851–863, (1967). MR0214489

- [15] D.J. Klein. Rigorous results for branched polymer models with excluded volume. *J. Chem. Phys.*, **75**:5186–5189, (1981).
- [16] P.J. Pearard and D.S. Gaunt. $1/d$ -expansions for the free energy of lattice animal models of a self-interacting branched polymer. *J. Phys. A: Math. Gen.*, **28**:6109–6124, (1995). MR1364786
- [17] M.D. Penrose. On the spread-out limit for bond and continuum percolation. *Ann. Appl. Probab.*, **3**:253–276, (1993). MR1202526
- [18] M.D. Penrose. Self-avoiding walks and trees in spread-out lattices. *J. Stat. Phys.*, **77**:3–15, (1994). MR1300525
- [19] G. Slade. *The Lace Expansion and its Applications*. Springer, Berlin, (2006). Lecture Notes in Mathematics Vol. 1879. Ecole d’Eté de Probabilités de Saint–Flour XXXIV–2004. MR2239599
- [20] R.P. Stanley. *Enumerative Combinatorics*, volume 1. Cambridge University Press, Cambridge, (1997). MR1442260